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# Mixed monotone operators and their application to integral equations

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**Abstract.** In this paper, we investigate the mixed monotone operators of a new type in a Cartesian product of a Banach space with an order determined by a normal cone. We establish sufficient conditions for such operators to have a unique fixed point and provide monotone iterative techniques which give sequences convergent to the fixed point. We also apply the obtained results to prove the existence and uniqueness of a solution of a nonlinear integral equation.

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**Keywords.** $(e, u)$-concave-convex operator, fixed point, mixed monotone operator, nonlinear integral equation, existence and uniqueness of a solution.

## 1. Introduction and preliminaries

The history of mixed monotone operators goes back to 1987 when Guo and Lakshmikantham [3] first obtained results concerning the application of nonlinear operators via coupled fixed points. Since then, many theorems in this field have been proved and many new applications have been presented; see, e.g., [2, 5, 4, 6, 1, 8]. It is also worthy to mention the recent paper [7] by Zhao, where he defined the $e$-concave-convex operator and investigated the existence and uniqueness of a fixed point for the operator of this type in the context of an ordered Banach space with different types of a cone determining the order. Zhao presented also the application of $e$-concave-convex operators to obtain the existence and uniqueness of a solution for a certain class of integral equations. Consequently, he improved the results of Wu and Liang [6].

The concept of the present work is twofold. First, we introduce a new concept of $(e, u)$-concave-convex operator, being a natural development of the operators investigated in [6, 7], show some of its useful properties and finally state a fixed point result for such operators. In the second part of the paper we present the application of the fixed points of $(e, u)$-concave-convex operators to a certain general class of nonlinear integral equations.

For the sake of convenience, first, we recall some useful notations and definitions.

Throughout the paper, $E$ is a real Banach space with a cone $P \subset E$. A cone $P$ determines the partial ordering in $E$: for $u, v \in E$, $u \leq v$ if and only if $v - u \in P$. A cone $P$ is said to be normal if there exists $N > 0$ such that for each $u, v \in E$, $\theta \leq u \leq v$, we have $\|u\| \leq N\|v\|$. Denote $P^+ = P - \{\theta\}$. In [6] Wu and Liang considered the following set:

$$C_e = \{x \in E\colon \alpha e \leq x \leq \beta e \text{ for some } \alpha, \beta > 0\}, \quad e \in P^+,$$

and investigated a $t - \alpha(t)$ mixed monotone operator, i.e., the operator $A\colon P_e \times P_e \to P_e$ satisfying

$$A\big(\lambda u, \lambda^{-1} v\big) \geq \lambda^{\alpha(\lambda)} A(u, v) \quad \text{for all } u, v \in P,\ 0 < \lambda < 1,$$

where $0 < \alpha(\lambda) < 1$ for all $\lambda \in (0, 1)$. The authors obtained fixed point results for such operators.

Next, Zhao [7] considered the $e$-concave-convex operator, i.e., the mapping $A\colon C_e \times C_e \to C_e$, $e \in P^+$, described by the following inequality:

$$\forall_{u,v \in C_e}\ \forall_{0<\lambda<1} \quad A\big(\lambda u, \lambda^{-1} v\big) \geq \lambda\big(1 + \eta(u, v, \lambda)\big) A(u, v),$$

where $\eta(u, v, \lambda) > 0$ for all $u, v \in C_e$, $0 < \lambda < 1$. In this way, Zhao generalized the results due to Wu and Liang [6].

Let $X \subset E$, $x, y \in X$ and let $A\colon X \times X \to E$ be a mapping. A pair $(x, y)$ is called a coupled quasi-fixed point of $A$ if $A(x, y) = x$ and $A(y, x) = y$. If $A(x, x) = x$, then $x$ is called a fixed point of $A$. The operator $A$ is called mixed monotone if $x_1 \leq x_2$, $y_2 \leq y_1$ imply $A(x_1, y) \leq A(x_2, y)$, $A(x, y_1) \leq A(x, y_2)$, respectively, for all $x, x_1, x_2, y, y_1, y_2 \in X$, i.e., $A$ is nondecreasing in its first argument and nonincreasing in its second argument.

## 2. The results

First, we introduce a new type of operator and show some of its important properties.

For $e \in P^+$, $u \in P$, $u \leq e$ put

$$C_{e,u} = \{x \in E\colon \alpha e \leq x + u \leq \beta e \text{ for some } \alpha, \beta > 0\}.$$

**Remark 2.1.** Obviously $C_{e,\theta} = C_e$ for each $e \in P^+$. Observe that $C_e \subset P^+$ for any $e \in P^+$, while the set $C_{e,u}$ need not be even a subset of the cone $P$ for some $e \in P^+$, $u \in P$, $u \leq e$. Therefore the sets $C_e$ and $C_{e,u}$ are of different nature.

**Definition 2.1.** Let $e \in P^+$, $u \in P$. An operator $A\colon C_{e,u} \times C_{e,u} \to C_{e,u}$ is called an $(e, u)$-concave-convex operator if there exists a mapping

$$L\colon C_{e,u} \times C_{e,u} \times (0, 1) \to (0, \infty)$$

such that for all $x, y \in C_{e,u}$ and $\lambda \in (0, 1)$ the following conditions hold:

(A1) $L(x, y, \lambda) > \lambda$;
(A2) $A(\lambda x+(\lambda-1)u, \lambda^{-1}y+(\lambda^{-1}-1)u) \geq L(x, y, \lambda)A(x, y)+(L(x, y, \lambda)-1)u$.

**Remark 2.2.** Observe that condition (A2) is well defined, i.e., for any $x, y \in C_{e,u}$ and $0 < \lambda < 1$, $\lambda x + (\lambda - 1)u$ and $\lambda^{-1}y + (\lambda^{-1} - 1)u$ are the elements of $C_{e,u}$. Indeed, taking $\alpha_1, \alpha_2, \beta_1, \beta_2 > 0$ such that

$$\alpha_1 e \leq x + u \leq \beta_1 e, \quad \alpha_2 e \leq y + u \leq \beta_2 e,$$

one can easily see that the following inequalities hold:

$$\lambda\alpha_1 e \leq \big(\lambda x + (\lambda - 1)u\big) + u \leq \lambda\beta_1 e,$$
$$\lambda^{-1}\alpha_2 e \leq \big(\lambda^{-1}y + (\lambda^{-1} - 1)u\big) + u \leq \lambda^{-1}\beta_2 e.$$

**Remark 2.3.** If we set in Definition 2.1 $u = \theta$ and $L(x, y, \lambda) = \lambda^{\alpha(\lambda)}$, where $0 < \alpha(\lambda) < 1$, $\lambda \in (0, 1)$, then we get a $t - \alpha(t)$ mixed monotone model operator due to Wu and Liang [6]. Moreover, observe also that taking in the above definition $u = \theta$ and $L(x, y, \lambda) = \lambda(1+\eta(x, y, \lambda))$, where $\eta(x, y, \lambda) > 0$, we obtain the definition of $e$-concave-convex operator defined by Zhao [7]. In the last section of the present paper we will show that such more general operators can be applied to prove the existence of a solution for a certain class of integral equations.

**Proposition 2.1.** *Every coupled quasi-fixed point of a mixed monotone $(e, u)$-concave-convex operator $A$ is a fixed point of $A$.*

*Proof.* Suppose that $(x^*, y^*)$ is a coupled quasi-fixed point of $A$. There exists $t_0 > 0$ such that $x^* \geq t_0 y^* + (t_0 - 1)u$. Therefore, a set $U$ of the form

$$U = \big\{t > 0 \colon x^* \geq ty^* + (t - 1)u\big\}$$

is not empty. Suppose that $U \subseteq (0, 1)$ and put

$$\tilde{t} = \max U. \tag{2.1}$$

Since $U$ is closed, we get $\tilde{t} < 1$. Thus, using (A2), we obtain

$$\begin{aligned} x^* = A(x^*, y^*) &\geq A\left(\tilde{t}y^* + \left(\tilde{t} - 1\right)u, \tilde{t}^{-1}x^* + \left(\tilde{t}^{-1} - 1\right)u\right) \\ &\geq L\left(y^*, x^*, \tilde{t}\right)y^* + \left(L\left(y^*, x^*, \tilde{t}\right) - 1\right)u, \end{aligned}$$

which gives $L(y^*, x^*, \tilde{t}) \in U$. Consequently, (2.1) and (A1) imply

$$\tilde{t} \geq L\left(y^*, x^*, \tilde{t}\right) > \tilde{t},$$

which is a contradiction. Therefore, there must exist $t \in U$, $t \geq 1$, and hence

$$x^* \geq ty^* + (t - 1)u \geq y^*.$$

Using analogous arguments, we show that $y^* \geq x^*$, which gives $x^* = y^*$. □

**Proposition 2.2.** *Every mixed monotone $(e, u)$-concave-convex operator $A$ has at most one fixed point.*

*Proof.* Assume that $A(x^*, x^*) = x^*$, $A(y^*, y^*) = y^*$ for some $x^*, y^* \in C_{e,u}$. Denote

$$D = \{t > 0\colon x^* \geq ty^* + (t-1)u,\ y^* \geq tx^* + (t-1)u\}.$$

There exist $r_0, s_0 > 0$ such that

$$x^* \geq r_0 y^* + (r_0 - 1)u, \qquad y^* \geq s_0 x^* + (s_0 - 1)u.$$

Take $t_0 = \min\{r_0, s_0\}$. We see that $D \neq \varnothing$ since $t_0 \in D$. Observe that if $t \in D$, then $t \leq 1$. Thus, due to the fact that $D$ is nonempty, bounded and closed, we can take

$$\tilde{t} = \max D. \tag{2.2}$$

Suppose that $\tilde{t} < 1$. From (A2) we get

$$\begin{aligned} x^* = A(x^*, x^*) &\geq A\left(\tilde{t}y^* + \left(\tilde{t} - 1\right)u, \tilde{t}^{-1}y^* + \left(\tilde{t}^{-1} - 1\right)u\right) \\ &\geq L\left(y^*, y^*, \tilde{t}\right)A(y^*, y^*) + \left(L\left(y^*, y^*, \tilde{t}\right) - 1\right)u \\ &= L\left(y^*, y^*, \tilde{t}\right)y^* + \left(L\left(y^*, y^*, \tilde{t}\right) - 1\right)u. \end{aligned} \tag{2.3}$$

By the same way we obtain

$$y^* \geq L\left(x^*, x^*, \tilde{t}\right)x^* + \left(L\left(x^*, x^*, \tilde{t}\right) - 1\right)u. \tag{2.4}$$

From (2.3) and (2.4), we have

$$k = \min\left\{L\left(x^*, x^*, \tilde{t}\right), L\left(y^*, y^*, \tilde{t}\right)\right\} \in D.$$

In consequence, by (A1) and (2.2) we obtain $\tilde{t} < k \leq \tilde{t}$, which is impossible, and hence $\tilde{t} = 1$, which gives $x^* = y^*$. $\square$

Denote

$$F = \left\{(x, y) \in C_{e,u} \times C_{e,u}\colon x \leq A(x, y), A(y, x) \leq y\right\}.$$

**Proposition 2.3.** *If a mixed monotone $(e, u)$-concave-convex operator $A$ has a fixed point $x^* \in C_{e,u}$, then*

$$x^* = \max_{x \in C_{e,u}} \left\{\exists_{y \in C_{e,u}} (x, y) \in F\right\} = \min_{y \in C_{e,u}} \left\{\exists_{x \in C_{e,u}} (x, y) \in F\right\}.$$

*Proof.* First, note that $F$ is nonempty, since $(x^*, x^*) \in F$. Take any $(x, y) \in F$.

There exist $\alpha_1, \beta_1, \alpha_2, \beta_2 > 0$ satisfying

$$\alpha_1 e \leq x^* + u \leq \beta_1 e, \quad \alpha_2 e \leq A(y, x) + u, \quad A(x, y) + u \leq \beta_2 e.$$

From the above we know that

$$\begin{aligned} rx^* + (r-1)u &\leq A(y, x) \leq y && \text{for some } r > 0, \\ sx + (s-1)u &\leq sA(x, y) + (s-1)u \leq x^* && \text{for some } s > 0. \end{aligned}$$

Taking $t = \min\{r, s\}$ we get

$$tx + (t-1)u \leq x^*, \quad tx^* + (t-1)u \leq y.$$

Therefore, the set $D$ of the form

$$D = \left\{t > 0\colon tx + (t-1)u \leq x^*,\ tx^* + (t-1)u \leq y\right\}$$

is not empty. Moreover, observe that $D$ is bounded. In other case it would exist a sequence $(t_n) \subset D$ of positive numbers such that $\lim_{n\to\infty} t_n = \infty$ and then we would have

$$t_n x + (t_n - 1)u \leq x^* \iff x + \left(1 - \frac{1}{t_n}\right)u \leq \frac{1}{t_n} x^*.$$

Letting $n \to \infty$ we obtain $x + u \leq \theta$, which is impossible.

From the above and the fact that $D$ is closed, we can put

$$\tilde{t} = \max D. \tag{2.5}$$

Obviously, $\tilde{t} > 0$. Suppose that $\tilde{t} < 1$. From (A2) and the fact that $\tilde{t} \in D$ and $(x, y) \in F$, we have

$$\begin{aligned} x^* = A(x^*, x^*) &\geq A\left(\tilde{t}x + \left(\tilde{t} - 1\right)u, \tilde{t}^{-1}y + \left(\tilde{t}^{-1} - 1\right)u\right) \\ &\geq L\left(x, y, \tilde{t}\right)A(x, y) + \left(L\left(x, y, \tilde{t}\right) - 1\right)u \\ &\geq L\left(x, y, \tilde{t}\right)x + \left(L\left(x, y, \tilde{t}\right) - 1\right)u \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} y &\geq A(y, x) \geq A\left(\tilde{t}x^* + \left(\tilde{t} - 1\right)u, \tilde{t}^{-1}x^* + \left(\tilde{t}^{-1} - 1\right)u\right) \\ &\geq L\left(x^*, x^*, \tilde{t}\right)A(x^*, x^*) + \left(L\left(x^*, x^*, \tilde{t}\right) - 1\right)u \\ &= L\left(x^*, x^*, \tilde{t}\right)x^* + \left(L\left(x^*, x^*, \tilde{t}\right) - 1\right)u. \end{aligned} \tag{2.7}$$

Formulas (2.6) and (2.7) imply that

$$k = \min\left\{L\left(x, y, \tilde{t}\right), L\left(x^*, x^*, \tilde{t}\right)\right\} \in D.$$

In consequence, by (A1) and (2.5) we obtain $\tilde{t} < k \leq \tilde{t}$, which is impossible and thus $\tilde{t} \geq 1$. Therefore, we get

$$x \leq \tilde{t}x + \left(\tilde{t} - 1\right)u \leq x^* \leq \tilde{t}x^* + \left(\tilde{t} - 1\right)u \leq y$$

and the proof is completed. □

**Proposition 2.4.** *If a mixed monotone $(e, u)$-concave-convex operator $A$ has a fixed point $x^* \in C_{e,u}$, then for any $x_0, y_0 \in C_{e,u}$ the sequences*

$$x_n = A(x_{n-1}, y_{n-1}), \quad y_n = A(y_{n-1}, x_{n-1}), \quad n \in \mathbb{N},$$

*satisfy*

$$\begin{aligned} &\alpha_n(x^* + u) \leq x_n - x^* \leq \beta_n(x^* + u), \\ &\alpha_n(x^* + u) \leq y_n - x^* \leq \beta_n(x^* + u), \quad n \in \mathbb{N}, \end{aligned}$$

*where $\alpha_n \nearrow 0$, $\beta_n \searrow 0$.*

*Proof.* Let $x^* \in C_{e,u}$ satisfy $A(x^*, x^*) = x^*$ and take any $x_0, y_0 \in C_{e,u}$. There exists $t_1 \in (0, 1)$ such that

$$t_1 x^* \leq x_0 + (1 - t_1)u, \quad x_0 \leq t_1^{-1}x^* + \left(t_1^{-1} - 1\right)u \tag{2.8}$$

and

$$t_1 x^* \leq y_0 + (1 - t_1)u, \quad y_0 \leq t_1^{-1}x^* + \left(t_1^{-1} - 1\right)u. \tag{2.9}$$

We set

$$v_0 = t_1 x^* + (t_1 - 1)u, \quad w_0 = t_1^{-1} x^* + \big(t_1^{-1} - 1\big)u,$$
$$v_n = A(v_{n-1}, w_{n-1}), \qquad w_n = A(w_{n-1}, v_{n-1}), \quad n \in \mathbb{N}.$$

Observe that $v_0 \le x^* \le w_0$ and $v_0 \le x_0 \le w_0$, $v_0 \le y_0 \le w_0$. Assume that for $k > 1$ the following inequality holds:

$$v_{k-1} \le x^* \le w_{k-1}.$$

We have

$$v_k = A(v_{k-1}, w_{k-1}) \le A(x^*, x^*) \le A(w_{k-1}, v_{k-1}) = w_k.$$

By the same way we can show, by induction, the following inequalities:

$$v_n \le x_n \le w_n, \quad v_n \le y_n \le w_n \quad \text{for all } n \in \mathbb{N}. \tag{2.10}$$

Moreover, observe that

$$\begin{aligned} v_1 &= A\big(t_1 x^* + (t_1 - 1)u, t_1^{-1} x^* + (t_1^{-1} - 1)u\big) \\ &\ge L(x^*, x^*, t_1)x^* + \big(L(x^*, x^*, t_1) - 1\big)u \\ &\ge t_1 x^* + (t_1 - 1)u = v_0. \end{aligned} \tag{2.11}$$

From (A2), for any $x, y \in C_{e,u}$ and $\lambda \in (0, 1)$, the following inequality holds:

$$\begin{aligned} A(x, y) \le\ & L(x, y, \lambda)^{-1} A\big(\lambda x + (\lambda - 1)u, \lambda^{-1} y + (\lambda^{-1} - 1)u\big) \\ & + \big(L(x, y, \lambda)^{-1} - 1\big)\, u. \end{aligned}$$

Hence, we get

$$\begin{aligned} w_1 &= A\big(t_1^{-1} x^* + (t_1^{-1} - 1)u, t_1 x^* + (t_1 - 1)u\big) \\ &\le L\big(t_1^{-1} x^* + (t_1^{-1} - 1)u, t_1 x^* + (t_1 - 1)u, t_1\big)^{-1} x^* \\ &\quad + \Big(L\big(t_1^{-1} x^* + (t_1^{-1} - 1)u, t_1 x^* + (t_1 - 1)u, t_1\big)^{-1} - 1\Big)u \\ &\le t_1^{-1} x^* + \big(t_1^{-1} - 1\big)u = w_0. \end{aligned} \tag{2.12}$$

Again, using induction, we obtain, for all $n \in \mathbb{N}$,

$$v_0 \le v_1 \le \cdots \le v_n \le \cdots \le w_n \le \cdots \le w_1 \le w_0. \tag{2.13}$$

For each $n \in \mathbb{N}$, let us denote

$$D_n = \big\{t > 0\colon tx^* + (t - 1)u \le v_n,\ w_n \le t^{-1} x^* + (t^{-1} - 1)u\big\}.$$

From (2.11) and (2.12) we have that $D_1 \ne \varnothing$. Formula (2.13) implies

$$D_n \subset D_{n+1} \quad \text{for all } n \in \mathbb{N}. \tag{2.14}$$

It is easy to see that each $D_n$ is bounded and closed, thus we can set

$$\alpha_n = \max D_n, \quad n \in \mathbb{N}. \tag{2.15}$$

From (2.14) we have that $(\alpha_n)$ is a nondecreasing sequence. Moreover, observe that $\alpha_n \le 1$, $n \in \mathbb{N}$. If $\alpha_{n_0} > 1$ for some $n_0 \in \mathbb{N}$, then

$$x^* < \alpha_{n_0} x^* + (\alpha_{n_0} - 1)u \le v_{n_0}$$

and

$$w_{n_0} \leq \alpha_{n_0}^{-1} x^* + \big(\alpha_{n_0}^{-1} - 1\big) u < x^*,$$

that implies $w_{n_0} < v_{n_0}$ which, due to (2.13), is impossible.

In the following we show that $\lim_{n\to\infty} \alpha_n = 1$. Suppose that

$$\lim_{n\to\infty} \alpha_n = \tilde{\alpha} < 1$$

and let $\eta > 0$ be such that $L(x^*, x^*, \tilde{\alpha}) = \tilde{\alpha} + \eta$. Then, for all $n \in \mathbb{N}$, we have

$$\begin{aligned}
v_{n+1} &= A(v_n, w_n) \geq A\big(\alpha_n x^* + (\alpha_n - 1)u, \alpha_n^{-1} x^* + \big(\alpha_n^{-1} - 1\big)u\big) \\
&= A\bigg(\frac{\alpha_n}{\tilde{\alpha}}\big[\tilde{\alpha} x^* + (\tilde{\alpha} - 1)\, u\big] + \bigg(\frac{\alpha_n}{\tilde{\alpha}} - 1\bigg)u, \\
&\qquad \frac{\tilde{\alpha}}{\alpha_n}\big[\tilde{\alpha}^{-1} x^* + \big(\tilde{\alpha}^{-1} - 1\big)\, u\big] + \bigg(\frac{\tilde{\alpha}}{\alpha_n} - 1\bigg)u\bigg) \\
&\geq L\bigg(\tilde{\alpha} x^* + (\tilde{\alpha} - 1)\, u, \tilde{\alpha}^{-1} x^* + \big(\tilde{\alpha}^{-1} - 1\big)\, u, \frac{\alpha_n}{\tilde{\alpha}}\bigg) \\
&\quad \times A\Big(\tilde{\alpha} x^* + (\tilde{\alpha} - 1)\, u, \tilde{\alpha}^{-1} x^* + \big(\tilde{\alpha}^{-1} - 1\big)\, u\Big) \\
&\quad + \bigg(L\bigg(\tilde{\alpha} x^* + (\tilde{\alpha} - 1)u, \tilde{\alpha}^{-1} x^* + (\tilde{\alpha}^{-1} - 1)u, \frac{\alpha_n}{\tilde{\alpha}}\bigg) - 1\bigg)u \\
&\geq L\bigg(\tilde{\alpha} x^* + (\tilde{\alpha} - 1)\, u, \tilde{\alpha}^{-1} x^* + \big(\tilde{\alpha}^{-1} - 1\big)\, u, \frac{\alpha_n}{\tilde{\alpha}}\bigg) \\
&\quad \times \big[L\left(x^*, x^*, \tilde{\alpha}\right) x^* + \left(L\left(x^*, x^*, \tilde{\alpha}\right) - 1\right) u\big] \\
&\quad + \bigg(L\bigg(\tilde{\alpha} x^* + (\tilde{\alpha} - 1)u, \tilde{\alpha}^{-1} x^* + (\tilde{\alpha}^{-1} - 1)u, \frac{\alpha_n}{\tilde{\alpha}}\bigg) - 1\bigg)u \\
&\geq \frac{\alpha_n}{\tilde{\alpha}}\big[\left(\tilde{\alpha} + \eta\right) x^* + \left(\tilde{\alpha} + \eta - 1\right) u\big] + \bigg(\frac{\alpha_n}{\tilde{\alpha}} - 1\bigg)u \\
&= \alpha_n\bigg(1 + \frac{\eta}{\tilde{\alpha}}\bigg)x^* + \bigg(\alpha_n + \frac{\alpha_n}{\tilde{\alpha}}\eta - 1\bigg)u.
\end{aligned}$$

Using analogous calculations, we obtain

$$w_{n+1} \leq \alpha_n^{-1}\bigg(1 + \frac{\varepsilon}{\tilde{\alpha}}\bigg)^{-1} x^* + \bigg(\alpha_n^{-1}\bigg(1 + \frac{\varepsilon}{\tilde{\alpha}}\bigg)^{-1} - 1\bigg)u,$$

where $\varepsilon > 0$ is such that

$$L\left(\tilde{\alpha}^{-1} x^* + \big(\tilde{\alpha}^{-1} - 1\big)\, u, \tilde{\alpha} x^* + (\tilde{\alpha} - 1)\, u, \tilde{\alpha}\right) = \tilde{\alpha} + \varepsilon.$$

Taking

$$\theta = \min\{1 + \varepsilon/\tilde{\alpha}, 1 + \eta/\tilde{\alpha}\},$$

we get that $\alpha_n \theta \in D_{n+1}$. Hence, due to (2.15), we obtain $\alpha_{n+1} \geq \alpha_n \theta$ for all $n \in \mathbb{N}$. Taking $n \to \infty$, we have $\tilde{\alpha} \geq \tilde{\alpha}\theta > \tilde{\alpha}$, which is impossible and hence $\lim_{n\to\infty} \alpha_n = 1$.

Finally, from (2.10), we get

$$\alpha_n x^* + (\alpha_n - 1)u \le x_n \le \alpha_n^{-1} x^* + \big(\alpha_n^{-1} - 1\big)u,$$
$$\alpha_n x^* + (\alpha_n - 1)u \le y_n \le \alpha_n^{-1} x^* + \big(\alpha_n^{-1} - 1\big)u.$$

In consequence,

$$(\alpha_n - 1)(x^* + u) \le x_n - x^* \le \big(\alpha_n^{-1} - 1\big)(x^* + u),$$
$$(\alpha_n - 1)(x^* + u) \le y_n - x^* \le \big(\alpha_n^{-1} - 1\big)(x^* + u),$$

which ends the proof. □

In the following we will use the notation

$$G^t_{v_0,w_0} = \left\{ \frac{L(u,v,t)}{t} : u, v \in [v_0, w_0] \right\}, \quad t > 0,\ v_0, w_0 \in C_{e,u},\ v_0 \le w_0.$$

**Theorem 2.1.** *Let $E$ be a real Banach space with a normal cone $P$, $e \in P^+$, $u \in P$, and let $A\colon C_{e,u} \times C_{e,u} \to C_{e,u}$ be a mixed monotone $(e,u)$-concave-convex operator. Suppose that there exist $v_0, w_0 \in C_{e,u}$, $v_0 \le w_0$, such that*

- (i) *$v_0 \le A(v_0, w_0)$, $A(w_0, v_0) \le w_0$;*
- (ii) *for any $\varepsilon \in (0,1)$ there exists $\delta \in (\varepsilon, 1)$ such that $\inf G^\delta_{v_0,w_0} > 1$.*

*Then $A$ has a fixed point in $[v_0, w_0]$.*

*Proof.* Denote

$$v_n = A(v_{n-1}, w_{n-1}), \quad w_n = A(w_{n-1}, v_{n-1}), \quad n \in \mathbb{N}.$$

From (i) we have $v_0 \le v_1 \le \cdots \le v_n \le \cdots \le w_n \le \cdots \le w_1 \le w_0$. Since $v_1, w_1 \in C_{e,u}$, there exists $t_1 > 0$ such that $v_1 \ge t_1 w_1 + (t_1 - 1)u$. Denote

$$E_n = \big\{ t > 0 \colon v_n \ge t w_n + (t-1)u \big\}.$$

Observe that for each $n \in \mathbb{N}$ we have

$$v_n \ge v_1 \ge t_1 w_1 + (t_1 - 1)u \ge t_1 w_n + (t_1 - 1)u,$$

and thus $E_n \ne \varnothing$. Observe that each $E_n$ is closed and bounded. Hence we can put

$$\eta_n = \max E_n, \quad n \in \mathbb{N}. \tag{2.16}$$

If $\eta_{n_0} > 1$ for some $n_0 \in \mathbb{N}$, then we get

$$v_{n_0} \ge \eta_{n_0} w_{n_0} + (\eta_{n_0} - 1)u > w_{n_0},$$

which is impossible. Thus $\eta_n \le 1$ for all $n \in \mathbb{N}$.

Observe that $\lim_{n\to\infty} \eta_n = 1$. Indeed, if $\lim_{n\to\infty} \eta_n = \tilde{\eta} < 1$, then from (ii) there exists $\delta \in (\tilde{\eta}, 1)$ such that

$$\varphi = \inf G^\delta_{u_0,v_0} > 1. \tag{2.17}$$

Next, for any $n \in \mathbb{N}$, we obtain

$$\begin{aligned}
v_{n+1} &= A(v_n, w_n) \geq A\left(\eta_n w_n + (\eta_n - 1)u, \eta_n^{-1} v_n + \left(\eta_n^{-1} - 1\right) u\right) \\
&= A\Bigg(\frac{\eta_n}{\delta}\left[\delta w_n + (\delta - 1)\, u\right] + \left(\frac{\eta_n}{\delta} - 1\right) u, \\
&\qquad\qquad \frac{\delta}{\eta_n}\left[\delta^{-1} v_n + \left(\delta^{-1} - 1\right) u\right] + \left(\frac{\delta}{\eta_n} - 1\right) u\Bigg) \\
&\geq L\left(\delta w_n + (\delta - 1)\, u, \delta^{-1} v_n + \left(\delta^{-1} - 1\right) u, \frac{\eta_n}{\delta}\right) \\
&\quad \times A\left(\delta w_n + (\delta - 1)\, u, \delta^{-1} v_n + \left(\delta^{-1} - 1\right) u\right) \\
&\quad + \left(L\left(\delta w_n + (\delta - 1)\, u, \delta^{-1} v_n + \left(\delta^{-1} - 1\right) u, \frac{\eta_n}{\delta}\right) - 1\right) u \\
&\geq \frac{\eta_n}{\delta}\left[L\left(w_n, v_n, \delta\right) A(w_n, v_n) + \left(L\left(w_n, v_n, \delta\right) - 1\right) u\right] + \left(\frac{\eta_n}{\delta} - 1\right) u \\
&= \eta_n \left[\frac{L\left(w_n, v_n, \delta\right)}{\delta} w_{n+1} + \left(\frac{L\left(w_n, v_n, \delta\right)}{\delta} - \frac{1}{\delta}\right) u\right] + \left(\frac{\eta_n}{\delta} - 1\right) u.
\end{aligned}$$

From the above and from (2.17) we get

$$\begin{aligned}
v_{n+1} &\geq \eta_n \left[\varphi w_{n+1} + \left(\varphi - \frac{1}{\delta}\right) u\right] + \left(\frac{\eta_n}{\delta} - 1\right) u \\
&= \eta_n \varphi w_{n+1} + (\eta_n \varphi - 1) u.
\end{aligned} \tag{2.18}$$

From (2.16) we know that

$$\eta_{n+1} = \sup\left\{t > 0 \colon v_{n+1} \geq t w_{n+1} + (t - 1) u\right\}.$$

Hence, by (2.18), we get

$$\eta_{n+1} \geq \eta_n \varphi \quad \text{for all } n \in \mathbb{N}.$$

Taking $n \to \infty$, we obtain $\tilde{\eta} \geq \tilde{\eta}\varphi > \tilde{\eta}$, which is impossible. Thus $\eta_n \to 1$, $n \to \infty$.

Now, consider any $m, n \in \mathbb{N}$, $m < n$. We have

$$\begin{aligned}
v_n - v_m \leq w_m - v_m &\leq w_m (1 - \eta_m) + (1 - \eta_m) u \\
&\leq w_0 (1 - \eta_m) + (1 - \eta_m) u.
\end{aligned}$$

From the normality of the cone, we have

$$\|v_n - v_m\| \leq K (1 - \eta_m) \|w_0 + u\|.$$

From the above we have that $(v_n)$ is a Cauchy sequence. Let $x^* \in [v_0, w_0]$ be such that $v_n \to x^*$. We have $v_n \leq x^* \leq w_n$ for all $n \in \mathbb{N}$. Thus we get

$$w_n - x^* \leq w_n - v_n \leq (1 - \eta_n)(w_0 + u),$$

and thus, by the normality of $P$, we have $w_n \to x^*$, $n \to \infty$.

Finally, for any $n \in \mathbb{N}$, we obtain

$$v_{n+1} = A(v_n, w_n) \le A(x^*, x^*) \le A(w_n, v_n) = w_{n+1}.$$

Therefore $x^*$ is a fixed point of $A$. $\square$

The next theorem is the main result of the paper.

**Theorem 2.2.** *Let $E$ be a real Banach space with a normal cone $P$, $e \in P^+$, $u \in P$, and let $A\colon C_{e,u} \times C_{e,u} \to C_{e,u}$ be a mixed monotone $(e,u)$-concave-convex operator. Suppose that the following conditions are satisfied:*

(L1) $\forall_{t>0}\ \forall_{u_1,u_2,v \in C_{e,u}}\ u_1 \le u_2 \Rightarrow L(u_1, v, t) \ge L(u_2, v, t)$;
(L2) $\forall_{t>0}\ \forall_{u,v_1,v_2 \in C_{e,u}}\ v_1 \le v_2 \Rightarrow L(u, v_1, t) \le L(u, v_2, t)$.

*Then $A$ has exactly one fixed point $x^* \in C_{e,u}$. Moreover, for any $x_0, y_0 \in C_{e,u}$, the sequences $x_n = A(x_{n-1}, y_{n-1})$, $y_n = A(y_{n-1}, x_{n-1})$, $n \in \mathbb{N}$, satisfy*

$$\|x_n - x^*\| \to 0, \quad \|y_n - x^*\| \to 0, \quad n \to \infty.$$

*Proof.* Since $A(e,e) \in C_{e,u}$, we can take $t_0 \in (0,1)$ such that

$$t_0 e + (t_0 - 1)u \le A(e,e) \le t_0^{-1} e + \big(t_0^{-1} - 1\big)u. \tag{2.19}$$

From (A1) we have $L(e,e,t_0)/t_0 > 1$. Thus we can take $n_0 \in \mathbb{N}$ such that

$$\frac{L(e,e,t_0)^{n_0}}{t_0^{n_0}} \ge \frac{1}{t_0}. \tag{2.20}$$

For all $n \in \mathbb{N}$ denote

$$a_n^+ = t_0^n e + \big(t_0^n - 1\big)u,$$
$$a_n^- = t_0^{-n} e + \big(t_0^{-n} - 1\big)u.$$

Note that for any $n \in \mathbb{N}$, by the fact that $t_0 \in (0,1)$ and by (L1), (L2), we have

$$L\big(a_n^+, a_n^-, t_0\big) \ge L(e,e,t_0). \tag{2.21}$$

Putting $v_0 = a_{n_0}^+$, $w_0 = a_{n_0}^-$, we have $v_0, w_0 \in C_{e,u}$ and $v_0 \le w_0$. Moreover, by (A2) and (2.21) we have

$$\begin{aligned}
A(v_0, w_0) &= A\big(a_{n_0}^+, a_{n_0}^-\big) = A\left(t_0 a_{n_0-1}^+ + (t_0 - 1)u, t_0^{-1} a_{n_0-1}^- + \big(t_0^{-1} - 1\big)u\right)\\
&\ge L\left(a_{n_0-1}^+, a_{n_0-1}^-, t_0\right) A\left(a_{n_0-1}^+, a_{n_0-1}^-\right)\\
&\quad + \left(L(a_{n_0-1}^+, a_{n_0-1}^-, t_0) - 1\right) u\\
&\ge L(e,e,t_0)\Big[L\left(a_{n_0-2}^+, a_{n_0-2}^-, t_0\right) A\left(a_{n_0-2}^+, a_{n_0-2}^-\right)\\
&\qquad\qquad + \left(L(a_{n_0-2}^+, a_{n_0-2}^-, t_0) - 1\right) u\Big] + \big(L(e,e,t_0) - 1\big)u\\
&\ge L(e,e,t_0)^2 A\left(a_{n_0-2}^+, a_{n_0-2}^-\right) + \left(L(e,e,t_0)^2 - 1\right) u\\
&\ge \cdots \ge L(e,e,t_0)^{n_0} A(e,e) + \left(L(e,e,t_0)^{n_0} - 1\right) u.
\end{aligned}$$

From the above, by (2.20) and (2.19) we get

$$A(v_0, w_0) \geq t_0^{n_0-1} A(e, e) + \left(t_0^{n_0-1} - 1\right) u \geq t_0^{n_0} e + \left(t_0^{n_0} - 1\right) u = v_0.$$

Conducting analogous argumentation, we can show that

$$A(w_0, v_0) \leq w_0.$$

Now, observe that for any $u, v \in [v_0, w_0]$ and $0 < t < 1$, we have

$$\frac{L(u, v, t)}{t} \geq \frac{L(w_0, v_0, t)}{t},$$

thus, by (A1)

$$\inf G^t_{v_0, w_0} \geq \frac{L(w_0, v_0, t)}{t} > 1.$$

Theorem 2.1 completes the proof. □

The immediate consequence of Theorem 2.2 are the following two corollaries.

**Corollary 2.1** [7, Theorem 3.1]**.** *Suppose that $E$ is a real Banach space with a normal cone $P$, $A\colon C_e \times C_e \to C_e$ is a mixed monotone and e-concave-convex operator. Assume that for any $t \in (0, 1)$, $\eta(u, v, t)$ is nonincreasing with respect to $u \in C_e$, nondecreasing with respect to $v \in C_e$. Then $A$ has exactly one fixed point $x^* \in C_e$. Moreover, constructing successively the sequences*

$$x_n = A(x_{n-1}, y_{n-1}), \quad y_n = A(y_{n-1}, x_{n-1}), \quad n = 1, 2, \ldots,$$

*for any initial values $x_0, y_0 \in C_e$, we have that*

$$\|x_n - x^*\| \to 0, \quad \|y_n - x^*\| \to 0 \quad \text{as } n \to \infty.$$

**Corollary 2.2** [6, Theorem 3.2]**.** *Suppose that $E$ is a real Banach space with a normal cone $P$, $e \in P^+$, $A\colon C_e \times C_e \to C_e$ is a mixed monotone operator. Assume that for all $0 < t < 1$ and $u, v \in C_e$ there exists*

$$0 < \alpha = \alpha(t, u, v) < 1$$

*such that*

$$A\left(tu, t^{-1} v\right) \geq t^{\alpha(t,u,v)} A(u, v).$$

*If $\alpha$ is nondecreasing in $u$ and nonincreasing in $v$, then there exists a unique point $x^* \in C_e$ such that $A(x,^*, x^*) = x^*$. Moreover, for any $x_0, y_0 \in C_e$, constructing successively the sequences*

$$x_n = A(x_{n-1}, y_{n-1}), \quad y_n = A(y_{n-1}, x_{n-1}), \quad n = 1, 2, \ldots,$$

*we have*

$$\|x_n - x^*\| \to 0, \quad \|y_n - x^*\| \to 0 \quad \text{as } n \to \infty.$$

## 3. Application

In this section, we apply our results in order to obtain the existence and uniqueness of a solution for a certain type of nonlinear integral equations.

Let $E$ be the set of all continuous bounded functions on $\mathbb{R}^N$. Together with the norm $\|x\| = \sup_{t\in\mathbb{R}^N} |x(t)|$, $E$ is a real Banach space with a normal cone of the form

$$P = \{x \in E\colon \forall_{t\in\mathbb{R}^N} x(t) \geq 0\}.$$

Take any $f \in P^+$ and $g \in P$, $g \leq f$, $g \neq f$.

Consider the following nonlinear integral equation:

$$\begin{aligned} x(t) + g(t) = \int_{\mathbb{R}^N} k(t,s) \Bigg[ & \sum_{i=1}^{N_1} \big(\alpha_i(s)x(s) + g(s)\big)^{a_i} \\ & + \sum_{j=1}^{N_2} \big(\beta_j(s)x(s) + g(s)\big)^{-b_j} \Bigg] ds, \end{aligned} \tag{3.1}$$

where $k\colon \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$, $\alpha_i\colon \mathbb{R}^N \to (0,1]$, $\beta_j\colon \mathbb{R}^N \to (0,1]$ are nonnegative and continuous functions, $a_i, b_j \in (0,1)$, $i = 1,2,\ldots, N_1, j = 1,2,\ldots, N_2, N_1, N_2 \in \mathbb{N}$. Moreover, assume that $k$ satisfies

$$\varphi(s)\big(f(t) + g(t)\big) \leq k(t,s) \leq \eta(s)\big(f(t) - g(t)\big) \quad \text{for all } t \geq 0, \tag{3.2}$$

where $\varphi\colon \mathbb{R}^N \to \mathbb{R}$ and $\eta\colon \mathbb{R}^N \to \mathbb{R}$ are nonnegative continuous and bounded functions, $\varphi, \eta \neq 0$.

**Theorem 3.1.** *Equation* (3.1) *has a unique solution in* $C_{f,g}$.

*Proof.* Consider the operator $A\colon C_{f,g} \times C_{f,g} \to E$ of the form

$$A(x,y)(t) = \int_{\mathbb{R}^N} k(t,s) h\big(s, x(s), y(s)\big)\, ds - g(t), \quad t \geq 0,$$

where $h\colon \mathbb{R}^N \times C_{f,g} \times C_{f,g} \to E$ is given by the formula

$$h\big(s, x(s), y(s)\big) = \sum_{i=1}^{N_1} \big(\alpha_i(s)x(s) + g(s)\big)^{a_i} + \sum_{j=1}^{N_2} \big(\beta_j(s)y(s) + g(s)\big)^{-b_j}.$$

Observe that the mapping $h$ is well defined, since taking any $s \in \mathbb{R}^N$ and $x, y \in C_{f,g}$ there exist $\alpha_1, \alpha_2 > 0$ such that

$$\alpha_1 f(s) \leq x(s) + g(s), \quad \alpha_2 f(s) \leq y(s) + g(s).$$

Thus, we get

$$\begin{aligned} \alpha_i(s)x(s) + g(s) &\geq \alpha_i(s)\big(\alpha_1 f(s) - g(s)\big) + g(s) \\ &= \alpha_1\alpha_i(s) f(s) + g(s)\big(1 - \alpha_i(s)\big) \\ &\geq 0, \quad i = 1,2,\ldots, N_1. \end{aligned}$$

By analogous calculations, we get

$$\beta_j(s)\,y(s) + g(s) \geq 0, \quad j = 1, 2, \ldots, N_2.$$

It is easy to observe that $A$ is a mixed monotone operator.

Now, we show that

$$A\big(C_{f,g} \times C_{f,g}\big) \subset C_{f,g}.$$

Let $x, y \in C_{f,g}$. By (3.2) we obtain

$$\begin{aligned} A(x,y)(t) + g(t) &= \int_{\mathbb{R}^{\mathbb{N}}} k(t,s) h\big(s, x(s), y(s)\big)\, ds \\ &\geq \int_{\mathbb{R}^{\mathbb{N}}} \big[\varphi(s)\big(f(t) + g(t)\big)\big] h\big(s, x(s), y(s)\big)\, ds \\ &\geq \int_{\mathbb{R}^{\mathbb{N}}} \varphi(s) h\big(s, x(s), y(s)\big)\, ds f(t), \quad t \geq 0, \end{aligned}$$

and

$$\begin{aligned} A(x,y)(t) + g(t) &= \int_{\mathbb{R}^{\mathbb{N}}} k(t,s) h\big(s, x(s), y(s)\big)\, ds \\ &\leq \int_{\mathbb{R}^{\mathbb{N}}} \eta(s)\big(f(t) - g(t)\big) h\big(s, x(s), y(s)\big)\, ds \\ &\leq \int_{\mathbb{R}^{\mathbb{N}}} \eta(s) h\big(s, x(s), y(s)\big) ds f(t), \quad t \geq 0. \end{aligned}$$

In order to show that $A$ is an $(f,g)$-concave-convex operator, take any $x, y \in C_{f,g}$ and $0 < \lambda < 1$. We get

$$\begin{aligned} &A\Big(\lambda x + (\lambda - 1)g, \lambda^{-1} y + \left(\lambda^{-1} - 1\right) g\Big)(t) \\ &= \int_{\mathbb{R}^N} k(t,s) \Bigg[ \sum_{i=1}^{N_1} \left[\alpha_i(s) \left(\lambda x(s) + (\lambda - 1) g(s)\right) + g(s)\right]^{a_i} \\ &\qquad + \sum_{j=1}^{N_2} \left[\beta_j(s) \left(\lambda^{-1} y(s) + (\lambda^{-1} - 1) g(s)\right) + g(s)\right]^{-b_j} \Bigg] ds - g(t) \\ &= \int_{\mathbb{R}^N} k(t,s) \Bigg[ \sum_{i=1}^{N_1} \lambda^{a_i} \left[\alpha_i(s) x(s) + \left(1 - \lambda^{-1}\right) \alpha_i(s) g(s) + \lambda^{-1} g(s)\right]^{a_i} \\ &\qquad + \sum_{j=1}^{N_2} \lambda^{b_j} \left[\beta_j(s) y(s) + (1 - \lambda) \beta_j(s) g(s) + \lambda g(s)\right]^{-b_j} \Bigg] ds - g(t) \\ &\geq \lambda^r \int_{\mathbb{R}^N} k(t,s) \Bigg[ \sum_{i=1}^{N_1} \left[\alpha_i(s) x(s) + \left(1 - \lambda^{-1}\right) \alpha_i(s) g(s) + \lambda^{-1} g(s)\right]^{a_i} \\ &\qquad + \sum_{j=1}^{N_2} \left[\beta_j(s) y(s) + (1 - \lambda) \beta_j(s) g(s) + \lambda g(s)\right]^{-b_j} \Bigg] ds - g(t), \end{aligned}$$

where

$$r = \max_{i=1,2,\dots,N_1, j=1,2,\dots,N_2} \{a_i, b_j\}.$$

Since $\alpha_i(s), \beta_j(s) \le 1$, $i = 1, 2, \dots, N_1$, $j = 1, 2, \dots, N_2$, $s \in \mathbb{R}^N$, we have

$$\begin{aligned}
&A\Big(\lambda x + (\lambda - 1)g, \lambda^{-1}y + \left(\lambda^{-1} - 1\right) g\Big)(t) \\
&\quad \ge \lambda^r \int_{\mathbb{R}^N} k(t,s) \Bigg[ \sum_{i=1}^{N_1} \big[\alpha_i(s)x(s) + g(s)\big]^{a_i} \\
&\qquad\qquad + \sum_{j=1}^{N_2} \big[\beta_j(s)y(s) + g(s)\big]^{-b_j} \Bigg] ds - g(t) \\
&\quad = \lambda^r \left[ \int_{\mathbb{R}^N} k(t,s)h\big(s, x(s), y(s)\big)\, ds - g(t) \right] + \lambda^r g(t) - g(t) \\
&\quad = L(x,y,\lambda)A(x,y)(t) + \big(L(x,y,\lambda) - 1\big)g(t),
\end{aligned}$$

where $L(x,y,\lambda) = \lambda^r$. Note that $\lambda^r > \lambda$, from which we get that $A$ is an $(f,g)$-concave-convex operator.

Finally, observe that (L1) and (L2) hold and, consequently, Theorem 2.2 ends the proof. □

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